ON THE HAMILTON-JACOBI METHOD IN VARIATIONAL PROBLEMS OF PARTIAL DIFFERENTIAL EQUATIONS

(O METODE GAMIL'TONA-IAKOBI V VARIATSIONNYKH ZADACHAKH S CHASTNYMI PROIZVODNYMI)

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Functionals of functions of many independent variables lie at the basis of a large class of problems of mathematical physics. Many works have been devoted to detailed studies of the corresponding variational problems. Of special importance among them are the investigations which extend the Hamilton-Jacobi method of integrating canonical systems to this class of problems. Some classical results along this line are due to Volterra [1, 2] and Fréchet [3] who obtained the first proofs for Jacobi's theorem. In [4] Lévy made a general study of equations in functional derivatives which play a fundamental role in the generalized Hamilton-Jacobi theory. These results, together with a number of others, have been presented in a systematic manner by Prange in [5].

The classical considerations are of geometric nature and they cannot be derived directly from the variational problem for many unknown functions of a single independent variable. Prange's results are somewhat different in this respect, even though they, too, were obtained by geometric means. A direct procedure similar to his, is, however, of great value in applications to the mechanics of continuous media. This procedure will be used in the present work.

For the sake of simplicity, we shall deal with problems for one unknown function of two independent variables. The generalization to the case of many unknown functions, or to more than two independent variables, does, seemingly, not present any serious difficulties.

1. Canonical system corresponding to the variational problem. Let us consider the problem of finding the extremum of the functional

$$I = \iint\limits_G L\,(x,\,y,\,z,\,z_x,\,z_y)\,dxdy \tag{1.1}$$

We impose the following condition on the comparison functions:

$$z^\circ = f(s) \qquad \text{on the boundary } \Gamma \text{ of the region } G \tag{1.2}$$

Here, and in what follows, the superscript $^\circ$ denotes values on the boundary.

Let us introduce the additional condition

$$\frac{\partial z}{\partial x} - z_x = 0, \qquad \frac{\partial z}{\partial y} - z_y = 0 \tag{1.3}$$

We shall refer to the equations (1.1) to (1.3) as the problem with three unknown functions z, z_x, z_y. With the aid of the Lagrange multipliers we construct the functional

$$\iint\limits_G \Big[L + p\left(\frac{\partial z}{\partial x} - z_x\right) + q\left(\frac{\partial z}{\partial y} - z_y\right)\Big]dxdy - \int\limits_\Gamma \rho\,(s)\ \left[z^\circ - f\,(s)\right] ds \tag{1.4}$$

Euler's equations have the form

$$L_{z_x} - p = 0, \quad L_{z_y} - q = 0, \quad L_z - \frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} = 0 \tag{1.5}$$

$$\frac{\partial z}{\partial x} - z_x = 0, \qquad \frac{\partial z}{\partial y} - z_y = 0 \tag{1.6}$$

The natural boundary condition

$$p^\circ \frac{\partial x}{\partial n} + q^\circ \frac{\partial y}{\partial n} - \rho = 0 \tag{1.7}$$

is obtained from (1.4) by integration by parts.

By selecting certain of the relations (1.5), (1.6) or (1.7) as the auxiliary conditions, we obtain different forms of the variational problem. If we take (1.6) for the auxiliary condition we arrive again at the problem (1.1) to (1.3). Adding the second of the conditions (1.5) and the first of (1.6), we obtain the canonical equations [5]

$$\frac{\partial z}{\partial y} = H_q, \qquad \frac{\partial q}{\partial y} = -H_z + \frac{\partial}{\partial x}\frac{\partial H}{\partial z_x} \tag{1.8}$$

or, making use of the symbol for the functional derivative, we have

$$\frac{\partial z}{\partial y} = \frac{\delta N}{\delta q}, \qquad \frac{\partial q}{\partial y} = -\frac{\delta N}{\delta z} \tag{1.9}$$

In order to obtain these equations, we introduce the notation

$$qz_y - L\ (x,\ y,\ z_x,\ z_y) = H\ (x,\ y,\ z,\ z_x,\ q) \tag{1.10}$$

namely, we perform the Legendre transformation with respect to the variable z_y.

The double integral in the expression (1.4) now takes on the form

$$\iint\limits_G \left[q\ \frac{\partial z}{\partial y} - H\ (x,\ y,\ z,\ z_x,\ q)\right] dx\,dy \tag{1.11}$$

The equations (1.9) are Euler's equations for the functional (1.11) of two functions z and q, or, what is the same thing, they are Hamilton's equations for the functional (1.1). In the sequel we shall refer to the system (1.9) as the canonical system.* In order to be able to write down the canonical equations one has to be able to solve the equations (1.5) for z_y.

We shall make use of the notation

$$M\ [z,\ z_y] = \int\limits_{x_0}^{x_1} L\ (x,\ y,\ z,\ z_x,\ z\ )\ dx, \qquad N\ [z,\ q] = \int\limits_{x_0}^{x_1} H\ (x,\ y,\ z,\ z_x,\ q)\ dx \tag{1.12}$$

The functionals M and N are connected by the relation

$$N = -\ M + \int\limits_{x_0}^{x_1} qz_y dx \tag{1.13}$$

2. The field of the functional (1.1). Prange [5] determines the field of the functional by a geometrical method. Following Gel'fand and Fomin [6], we will give a different determination of the field by making use of the concept of selfadjoint and matching boundary conditions.

The boundary conditions of the variational problem under consideration can be given in various ways. Of special interest are the boundary

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* Selecting the first two conditions of (1.5) for the auxiliary ones, one can obtain a different form for the canonical system, namely

$$\frac{\partial z}{\partial x} - \Phi_p = 0, \qquad \frac{\partial z}{\partial y} - \Phi_q = 0, \qquad \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \Phi_z = 0$$

$$\Phi\ (x,\ y,\ z,\ p,\ q) = pz_x + qz_y - L\ (x,\ y,\ z,\ z_x,\ z_y), \quad p = L_{z_x}, \qquad q = L_{z_y}$$

This form is used in Volterra [1, 2].

conditions which are determined by the functional itself in the following manner. Let $S = S[z;\ y]$ be the functional; z is the functional argument, and y plays the role of a parameter.

Let us consider the variational problem for the functional

$$\iint_G L(x, y, z, z_x, z_y)\,dxdy - S^{(1)}[z;\ a] + S^{(2)}[z;\ b] \tag{2.1}$$

where we impose no auxiliary conditions on the comparison functions. The functionals $S^{(1)}$ and $S^{(2)}$ are not the same in general. Natural boundary conditions of this problem are the following

$$\frac{\delta M}{\delta z_y} - \frac{\delta S^{(1)}}{\partial z}\Big|_{y=a} = 0, \qquad \frac{\delta M}{\delta z_y} - \frac{\delta S^{(2)}}{\delta z}\Big|_{y=b} = 0 \tag{2.2}$$

In place of $\delta M/\delta z_y$ we write $q(x, y, z, z_x, z_y)$ in accordance with the second condition (1.5). We shall also write S in place of $S^{(1)}$, and we consider the end condition

$$q(x, y, z, z_x, z_y) = \frac{\delta S}{\delta z}\Big|_{y=a} \tag{2.3}$$

At each point x of the straight line $y = a$, this condition gives the quantity $z_y(x, a)$ which is proportional to the corresponding direction cosine of the normal to the integral surface $z = z(x, y)$. It is convenient to write (2.3) in the form

$$z_y(x, a) = \Psi[z] \tag{2.4}$$

where $\Psi[z]$ is some functional.

The definitions and theorems which follow in this section are the direct generalizations of the corresponding facts of the calculus of variation in one variable as given in [6]. For the sake of completeness we considered it necessary to give them here with detailed proofs which, of course, are obtainable from the corresponding considerations in [6] by formally taking the limit of an infinite number of unknown functions.

Definition 2.1. The boundary conditions (2.4) ascribed to the functional (1.1) are called selfadjoint if there exists a functional $S[z, y]$ such that

$$\frac{\delta M[z, \Psi[z]]}{\delta z_y} \equiv \frac{\delta S}{\delta z}\Big|_{y=a} \tag{2.5}$$

We have the next theorem which is the analogue of Theorem 1 ([6], p.137).

Theorem 2.1. In order that the conditions (2.3) may be selfadjoint on the straight line $y = a$, it is necessary and sufficient that the following conditions hold

$$\frac{\delta q\,(x,\ a,\ z\,(x,\ a),\ z_x\,(x,\ a),\ \Psi\,[z\,(x,\ a)])}{\delta z\,(x_2,\ a)}\Bigg|_{x=x_1} = \\ = \frac{\delta q\,(x,\ a,\ z\,(x,\ a),\ z_x\,(x,\ a),\ \Psi\,[z\,(x,\ a)])}{\delta z\,(x_1,\ a)}\Bigg|_{x=x_2} \tag{2.6}$$

Proof. Necessity. The selfadjoint boundary conditions are defined by the equation (2.3). Taking it with $x = x_1$, and changing $z(x_2,\ a)$, or, conversely, fixing $x = x_2$, and changing $z(x_1,\ a)$, we obtain (2.6) because the right-hand sides of both results will hereby have the same value

$$\frac{\delta^2 S}{\delta z\,(x_1,\ a)\,\delta z\,(x_2,\ a)}$$

Sufficiency. If the conditions (2.4) imply (2.6), then there exists a functional $S[z]$ whose functional derivative at $y = a$, coincides with q. The variation of the functional

$$\iint\limits_G L dx dy - S\,|_{y=a}$$

set equal to zero, will then yield condition (2.3). This proves selfadjointness.

Definition 2.2. The boundary condition

$$z_y\,(x) = \Psi^{(1)}\,[z] \tag{2.7}$$

given when $y = a$, and the boundary condition

$$z_y\,(x) = \Psi^{(2)}\,[z] \tag{2.8}$$

given when $y = b$, are called matching conditions if every solution of the systems (1.9) satisfying the condition (2.7) with $y = a$ also satisfies the condition (2.8) with $y = b$, and conversely.

Definition 2.3. Suppose that for all $y (a \leqslant y \leqslant b)$ we are given the boundary conditions

$$z_y\,(x,\ y) = \Psi\,[z;\ y]$$

These boundary conditions form the field of the functional (1.1) if:

a) for every value of y the conditions are selfadjoint,

b) if for any pair y_1, y_2 from $[a, b]$ the conditions are matching.

Suppose that the selfadjointness of the boundary conditions has been established. What additional requirements must be imposed on these conditions to make them matching conditions? In other words, when will selfadjoint boundary conditions form a field of the functional (1.1)?

Theorem 2.2. The boundary conditions

$$\frac{\delta M}{\delta z_y} = \frac{\delta S}{\delta z} \tag{2.9}$$

will be matching conditions if the functional $S[z; y]$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial y} + \int_{x_0}^{x_1} H\left(x, y, z, z_x, \frac{\delta S}{\delta z}\right) dx = 0, \quad \text{or} \quad \frac{\partial S}{\partial y} + N\left[y, z; \frac{\delta S}{\delta z}\right] = 0 \tag{2.10}$$

This is a necessary and sufficient condition.

We shall show that under the conditions of this theorem every manifold in the space (x, y, z) on which (2.9) is satisfied is an integral surface of Euler's equation for the initial variational problem.* Equation (2.9) implies

$$q\,(x, y, z, z_x, \Psi\,[z]) = \frac{\delta S}{\delta z}$$

The equation (2.10) can, therefore, be rewritten as

$$\frac{\partial S}{\partial y} = -N\,[y, z; q]$$

Varying the last equation with respect to z, we obtain**

$$\frac{\partial}{\partial y}\frac{\delta S}{\delta z} = -\frac{\delta N\,[y, z;\, \Psi\,[z]]}{\delta z}, \quad \text{or} \quad \frac{\partial q}{\partial y} = -\frac{\delta N\,[y, z;\, \Psi\,[z]]}{\delta z} \tag{2.11}$$

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* The Hamilton-Jacobi equation can be obtained in the traditional way from Schroedinger's equation of the quantum field theory [7] by taking the "classical limit."

In recent years, Arzhanykh [8-10] has attempted to construct the Hamilton-Jacobi equation of the classical field theory; this equation is similar to the type considered by Volterra [1,2].

** We retain for the functional N the old notation in the next two formulas.

It follows from this that Euler's equation is satisfied. Indeed, in (2.11) let us replace q by $\delta M[z;\ \Psi[z]]/\delta z_y$ and the functional N by the expression (1.13); performing now the indicated operations in (2.11) and taking into account the equation $\delta M/\delta z_y = \partial L/\partial z_y$, we obtain

$$\frac{\partial}{\partial y}\frac{\delta M}{\delta z_y} + \int\limits_{x_0}^{x_1} \frac{\delta^2 M[z;\ \Psi[z]]}{\delta z_y(x,\ y)\,\delta z_y(\xi,\ y)}\,\frac{\partial \Psi[z(\xi,\ y);\ y]}{\partial y}\,d\xi = \frac{\delta M}{\delta z} + \int\limits_{x_0}^{x_1} \frac{\delta M[z;\ \Psi[z]]}{\delta z_y(\xi,\ y)}\,\frac{\delta \Psi[z(\xi,\ y);\ y]}{\delta z(x,\ y)}\,d\xi - \tag{2.12}$$
$$- \int\limits_{x_0}^{x_1} \frac{\delta \Psi[z(\xi,\ y);\ y]}{\delta z(x,\ y)}\,\frac{\delta M[z;\ \Psi[z]]}{\delta z_y(\xi,\ y)}\,d\xi - \int\limits_{x_0}^{x_1} \Psi[z(\xi,\ y);\ y]\,\frac{\delta^2 M[z;\ \Psi[z]]}{\delta z_y(\xi,\ y)\,\delta z(x,\ y)}\,d\xi$$

The condition of selfadjointness (2.9) yields

$$\frac{\delta^2 M[z;\ \Psi[z]]}{\delta z_y(\xi,\ y)\,\delta z(x,\ y)} = \frac{\delta^2 M[z;\ \Psi[z]]}{\delta z_y(x,\ y)\,\delta z(\xi,\ y)}$$

Therefore, equation (2.12) may be rewritten as

$$\frac{\delta M}{\delta z} = \frac{\partial}{\partial y}\frac{\delta M}{\delta z_y} + \int\limits_{x_0}^{x_1} \Psi[z(\xi,\ y);\ y]\,\frac{\delta^2 M[z;\ \Psi[z]]}{\delta z_y(x,\ y)\,\delta z(\xi,\ y)}\,d\xi + \int\limits_{x_0}^{x_1} \frac{\delta^2 M[z;\ \Psi[z]]}{\delta z_y(x,\ y)\,\delta z_y(\xi,\ y)}\,\frac{\partial \Psi[z(\xi,\ y);\ y]}{\partial y}\,d\xi \tag{2.13}$$

Since

$$\frac{\delta^2 M[z,\ z_y]}{\delta z(\xi,\ y)\,\delta z_y(x,\ y)} = \frac{\delta^2 M}{\delta z_y(x,\ y)\,\delta z(\xi,\ y)} + \int\limits_{x_0}^{x_1} \frac{\delta^2 M}{\delta z_y(x, y)\,\delta z_y(\zeta,\ y)}\,\frac{\delta \Psi[z(\zeta,\ y);\ y]}{\delta z(\xi,\ y)}\,d\zeta$$

equation (2.13) may be written in the form

$$\frac{\delta M}{\delta z} = \frac{\partial}{\partial y}\frac{\delta M}{\delta z_y} + \int\limits_{x_0}^{x_1} \frac{\delta^2 M}{\delta z_y(x,\ y)\,\delta z(\xi,\ y)}\,\Psi[z(\xi,\ y);\ y]\,d\xi + \tag{2.14}$$
$$+ \int\limits_{x_0}^{x_1} \frac{\delta^2 M}{\delta z_y(x,\ y)\,\delta z_y(\xi,\ y)}\left[\frac{\partial \Psi[z(\xi,\ y);\ y]}{\partial y} + \int\limits_{x_0}^{x_1} \Psi[z(\zeta,\ y);\ y]\,\frac{\delta \Psi[z(\xi,\ y);\ y]}{\delta z(\zeta,\ y)}\,d\zeta\right] d\xi$$

Taking into consideration $z_y = \Psi[z(x,\ y);\ y]$ and its implication

$$z_{yy} = \frac{\partial \Psi [z (x, y); y]}{\partial y} + \int_{x_0}^{x_1} \Psi [z (\zeta, y); y] \frac{\delta \Psi [z (x, y); y]}{\delta z (\zeta, y)} d\zeta$$

we can put equation (2.14) into the form

$$\frac{\delta M}{\delta z} = \frac{\partial}{\partial y} \frac{\delta M}{\delta z_y} + \int_{x_0}^{x_1} \frac{\delta^2 M}{\delta z_y (x, y) \delta z (\xi, y)} z_y (\xi, y) d\xi + \\ + \int_{x_0}^{x_1} \frac{\delta^2 M}{\delta z_y (x, y) \delta z_y (\xi, y)} z_{yy} (\xi, y) d\xi$$

or

$$\frac{\delta M}{\delta z} - \frac{d}{dy} \frac{\delta M}{\delta z_y} = 0 \tag{2.15}$$

Thus we have obtained Euler's equation of the original variational problem. This proves the sufficiency of the condition (2.10). Its necessity is established by reversing the above process.

The close analogy which exists between the one-dimensional and multidimensional variational problems is due to the fact that Jacobi's theorem, and with it the Hamilton-Jacobi method for integrating the canonical system, can be extended in a natural way to variational problems involving multiple integrals [1-5].

As is known, Jacobi's theorem in its classical formulation yields a method for obtaining the general solution of a canonical system by means of a known complete integral of the Hamilton-Jacobi equation.

If one considers a system with an infinite number of degrees of freedom, the behavior of which is described by the functional (1.1), as the limiting case of a holonomic system with a finite number (n) of degrees of freedom (for which Jacobi's classical theorem is valid) then there arises quite naturally the question: what becomes of the complete integral of the classical Hamilton-Jacobi equation in the limit as n goes to infinity?

The limiting form of this equation has already been obtained; it is equation (2.10). Its solution is the functional S. This functional coincides with the one which one obtains by the limit process from the complete integral of the corresponding finite-dimensional problem; it is natural to call it the complete Lévy [4] integral. The complete Lévy integral in the two-dimensional variational problem has two functional arguments: an unknown function $z(x, y)$, and a parametric function $\alpha(x)$;

besides that, it depends also on the variable y. In the *principal* functional of Lévy the parametric function coincides with the initial function $z(x, a)$. One should note that the Hamilton-Jacobi equation (2.10) has (in analogy with the classical one) many complete integrals. Here, we shall not dwell on the relationship among them. In particular, the complete Lévy integral does not always contain explicitly the parametric function $\alpha(x)$ as a functional argument. This function can appear in the integral, for example, in the form of an infinite number of constants. The parametric function appeared in the complete functional as the result of the limit process applied to a system of constants that were included in the complete integral of the Hamilton-Jacobi equation for the finite-dimensional "approximating" problem. It may happen that some of these constants become "isolated" constants also in the limit. The possibilities which can arise hereby are best considered on particular examples.

3. Hamilton-Jacobi method. We shall generalize a known inequality for the determinant of the matrix of mixed derivatives which characterizes the complete integral of the ordinary Hamilton-Jacobi equation.

The functional

$$S\,[z, \alpha; y] = \lim_{n\to\infty} S_n\,(z_1, \ldots, z_n, \alpha_1, \ldots, \alpha_n; y)$$

will be a complete Lévy integral of the equation (2.10) if

$$\lim_{n\to\infty} \left| \left\| \frac{\partial^2 S_n}{\partial z_i \partial \alpha_k} \right\| \right| \neq 0$$

The continuous analogue of this inequality is the sought condition. What is actually needed is that the equation (3.1) be solvable for z (see below).

The Hamilton-Jacobi method of solving the canonical system (1.9) is based on the validity of the following generalization of Jacobi's theorem.

Theorem. Let $S[z, \alpha; y]$ be a complete Lévy integral of the Hamilton-Jacobi equation (2.10), and let $\beta(x)$ be an arbitrary function. Then the functional $z = z[x, y; \alpha, \beta]$ which is determined by the equation*

$$\frac{\delta S}{\delta \alpha} = \beta \tag{3.1}$$

* The remark made above (p. 385) with respect to the parametric function applies to this formula.

and the functional

$$q = \frac{\delta S}{\delta z} \tag{3.2}$$

constitute the general solution of the canonical system (1.9).

The proof is immediate if one shows that on each integral surface

$$\frac{\partial}{\partial y}\frac{\delta S}{\delta \alpha} = 0$$

We have

$$\frac{\partial}{\partial y}\frac{\delta S}{\delta \alpha} = \frac{\partial}{\partial y}\frac{\delta S}{\delta \alpha} + \int\limits_{x_0}^{x_1} \frac{\delta^2 S}{\delta z(\xi, y)\,\delta\alpha} z_y(\xi, y)\,d\xi$$

Substituting $S = S[z, \alpha; y]$ in equation (2.10), and taking the variation with respect to α, we obtain

$$\frac{\partial}{\partial y}\frac{\delta S}{\delta \alpha} = -\int\limits_{x_0}^{x_1} \frac{\delta N}{\delta q(\xi, y)}\,\frac{\delta^2 S}{\delta z(\xi, y)\,\delta\alpha}\,d\xi$$

Substituting this into the preceding equation we see that

$$\frac{\partial}{\partial y}\left(\frac{\delta S}{\delta \alpha}\right) = \int\limits_{x_0}^{x_1} \frac{\delta^2 S}{\delta z(\xi, y)\,\delta\alpha}\left[z_y(\xi, y) - \frac{\delta N}{\delta q(\xi, y)}\right] d\xi$$

On the integral surface $z_y = \delta N/\delta q$; hence

$$\frac{\partial}{\partial y}\frac{\delta S}{\delta \alpha} = 0$$

which was to be proved.

If S is a complete Lévy integral then equation (3.1) can be solved for z and we thus obtain the problem's general solution which depends on two arbitrary functions α and β. The equation (3.2) determines then the canonical adjoint impulse q. The functions α and β are determined by the initial conditions.

4. Example. Let us consider as an example the simplest problem of the vibration of a string.

The equation $z_{yy} - z_{xx} = 0$ is Euler's equation for the functional

$$I = \frac{1}{2}\int\limits_0^y dy \int\limits_{-\infty}^{\infty} dx\,(z_y^2 - z_x^2)$$

We shall seek the solution of Cauchy's problem $z|_{y=0} = \alpha(x)$, $z_y|_{y=0} = \beta(x)$ by the use of the Hamilton-Jacobi method. Hamilton's function is given by

$$N = \frac{1}{2}\int_{-\infty}^{\infty} dx\,(z_y^2 + z_x^2)$$

and the equation (2.10) takes on the form

$$\frac{\partial S}{\partial y} + \frac{1}{2}\int_{-\infty}^{\infty} dx\left[\left(\frac{\delta S}{\delta z}\right)^2 + z_x^2\right] = 0$$

The finite-dimensional analogue is the Hamilton-Jacobi equation of a system of elastically connected oscillators separated by a distance a from each other. The functional $S[z;\ y]$ is here replaced by the function $S(z_1,\ \ldots,\ z_n;\ y)$ of $n+1$ variables (n is the number of oscillators)

$$\frac{\partial S}{\partial y} + \frac{1}{2}\sum_{i=1}^{n} a\left[\left(\frac{\partial S}{\partial z_i}\right)^2 + \left(\frac{z_{i+1} - z_i}{a}\right)^2\right] = 0$$

The complete integral of this equation is easily found by a transformation to the principal coordinates with the aid of the canonical transformation

$$\theta_i = \frac{2}{n+1}\sum_{\mu=1}^{n} z_\mu \sin\frac{i\mu\pi}{n+1}$$

The "old" impulses q_i will be connected with the "new" ones π_i by the relations

$$q_i = \frac{2}{n+1}\sum_{s=1}^{n} \pi_s \sin\frac{is\pi}{n+1}$$

In terms of the principal coordinates we have

$$S = a\sum_{i=1}^{n}\int\left[\left(\frac{E}{n} + \tau_i\right)\frac{2}{a} - \frac{\lambda_i^2\theta_i^2}{a^2}\right]^{1/2} d\theta_i - Ey \qquad \left(\sum_{i=1}^{n}\tau_i = 0\right)$$

Here E and τ_i are constants.

By Jacobi's classical theorem

$$y - y_0 = \frac{1}{n}\sum_{i=1}^{n}\int\left[\frac{2}{a}\left(\frac{E}{n} + \tau_i\right) - \frac{\lambda_i^2\theta_i^2}{a^2}\right]^{-1/2} d\theta_i$$

$$\beta_i = \frac{\partial S}{\partial \tau_i} = \int\left[\frac{2}{a}\left(\frac{E}{n} + \tau_i\right) - \frac{\lambda_i^2\theta_i^2}{a^2}\right]^{-1/2} d\theta_i - \int\left[\frac{2}{a}\left(\frac{E}{n} + \tau_n\right) - \frac{\lambda_n^2\theta_n^2}{a^2}\right]^{-1/2} d\theta_n$$

$$\pi_i = \frac{\partial S}{\partial \theta_i} = a\left[\left(\frac{E}{n} + \tau_i\right)\frac{2}{a} - \frac{\lambda_i^2\theta_i^2}{a^2}\right]^{1/2} \qquad \left(\lambda_i = 2\sin\frac{i\pi}{n+1}\right)$$

From these equations one obtains without difficulty

$$\theta_i = -\frac{1}{\lambda_i}\left[2a\left(\frac{E}{n}+\tau_i\right)\right]^{1/2}\sin\frac{\lambda_i}{a}(y-y_i), \qquad y_i = y_0 - \frac{1}{n}\sum_{k=1}^{n}{}' \beta_k$$

The superscript prime in the last sum indicates that the term β_i is to be omitted. For z_μ we obtain

$$z_\mu = \sum_{i=1}^{n}\theta_i \sin\frac{i\mu\pi}{n+1} = -\sum_{i=1}^{n}\frac{1}{\lambda_i}\left[2a\left(\frac{E}{n}+\tau_i\right)\right]^{1/2}\sin\frac{i\mu\pi}{n+1}\sin\frac{\lambda_i}{a}(y-y_i)$$

Let us take the limit as $n = \infty$, $a = 0$, $na = l$ (the string is fastened at the ends). We obtain

$$\lim_{n\to\infty}\frac{\lambda_i}{a} = \frac{i\pi}{l}, \qquad x = \frac{il}{n+1}, \qquad \frac{E}{na} = e, \qquad \lim_{n\to\infty}\frac{\tau_i}{a} = \sigma_i$$

$$\lim_{n\to\infty} z_i = z(x, y) = -\sum_{\mu=1}^{\infty}\frac{l}{\mu\pi}\sqrt{2(e+\sigma_\mu)}\sin\frac{\mu\pi x}{l}\sin\frac{\mu\pi}{l}(y-y_\mu)$$

From this one can easily obtain the next expression if one makes use of the initial conditions

$$z = \frac{2}{l}\int_0^l \sum_{\mu=1}^{\infty}\sin\frac{\mu\pi x}{l}\sin\frac{\mu\pi\xi}{l}\left[\alpha(\xi)\cos\frac{\mu\pi y}{l} + \beta(\xi)\frac{\sin(\mu\pi y)/l)}{\mu\pi/l}\right] d$$

This leads to the ordinary d'Alembert formula* if $l \to \infty$.

Let us write down the continuous analogues of the performed operations. For the infinite string the complete Lévy integral takes the form of the functional integral [11]

$$S = \int_{-\infty}^{\infty} d\xi \int^{\theta(\xi)} \sqrt{2[E\delta(\xi)+\sigma(\xi)] - \xi^2\theta'^2(\xi)}\, d\theta(\xi) - yE\int_{-\infty}^{\infty}\delta(\xi)d\xi$$

The lower limit of the inner integral is unessential; we shall set it equal to zero. In place of $\theta(\xi, y)$ we shall write $\theta(\xi)$ for the sake of brevity. The function $\delta(\xi)$ is subjected only to the condition

$$\int_{-\infty}^{\infty}\delta(\xi)\,d\xi = 1$$

and is arbitrary otherwise. The function $\sigma(\xi)$ is a parametric function satisfying the condition

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* With a corresponding change in the origin of the coordinate x.

$$\int_{-\infty}^{\infty} \sigma(\xi)\, d\xi = 0 \tag{4.1}$$

which is a generalization of the corresponding condition for the constants τ_i in the finite-dimensional problem.

Varying S with σ under the additional condition (4.1), we find

$$\beta(\xi) = \int_0^{\theta(\xi)} \frac{d\theta(\xi)}{\sqrt{2[E\delta(\xi)+\sigma(\xi)] - \xi^2\theta^2(\xi)}} + \lambda \tag{4.2}$$

Here λ is a Lagrange multiplier. The integral in this formula is taken in the function space θ along some path which connects the points 0 and $\theta(\xi)$. This integral can, however, be reduced quite easily to an ordinary integral [11]. In this connection we note that the integral (4.2) does not depend on the path of integration since the following condition is satisfied [11]

$$\frac{\delta}{\delta\theta(y)} \frac{1}{\sqrt{2[E\delta(x)+\sigma(x)] - x^2\theta^2(x)}} = \frac{\delta}{\delta\theta(x)} \frac{1}{\sqrt{2[E\delta(y)+\sigma(y)] - y^2\theta^2(y)}}$$

Let the integration path be the straight line $\theta = \theta(\xi)t$. Then $d\theta = \theta(\xi)dt$, and we obtain

$$\beta(\xi) = -\frac{1}{\xi} \sin^{-1} \left\{\frac{-\xi\theta(\xi)}{\sqrt{2[E\delta(\xi)+\sigma(\xi)]}}\right\} + \lambda \tag{4.3}$$

On the other hand, differentiating the complete Lévy integral with respect to E, we obtain

$$y - y_0 = \int_{-\infty}^{\infty} d\xi \int_0^{\theta(\xi)} \frac{\delta(\xi)\, d\theta(\xi)}{\sqrt{2[E\delta(\xi)+\sigma(\xi)] - \xi^2\theta^2(\xi)}}$$

Whence, as above

$$y - y_0 = -\int_{-\infty}^{\infty} \frac{\delta(\xi)\, d\xi}{\xi} \sin^{-1} \left\{-\frac{\xi\theta(\xi)}{\sqrt{2[E\delta(\xi)+\sigma(\xi)]}}\right\}$$

From this and (4.3) it follows that

$$y - y_0 = \int_{-\infty}^{\infty} \beta(\xi)\,\delta(\xi)\, d\xi - \lambda$$

Hence

$$\lambda = \int_{-\infty}^{\infty} \beta(\xi)\,\delta(\xi)\, d\xi - (y - y_0) \tag{4.4}$$

Now we determine $\theta(\xi)$ from (4.3) and (4.4). The remaining steps are obvious. It is easy to prove that the condition (4.1) is actually equivalent to the requirement that the initial data be regular at infinity.

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